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EULER-TYPE NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS WITH THREE LIOUVILLE FRACTIONAL DERIVATIVES [1]

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Abstract

This paper deals with the study of linear non-homogeneous ordinary differential equations with three right-hand sided Liouville derivatives of fractional order. Using the direct and inverse Mellin transforms and the residue theory, explicit solutions of the considered equations are established in terms of the generalized Wright functions, of the generalized hypergeometric functions and of the Euler psi-function. The corresponding results are deduced for ordinary differential equations of Euler type. Examples are given.

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1. Introduction

The paper is devoted to the solution in closed form of the linear non-homogeneous differential equations

$$\delta x^{\alpha+2}\left(D_{-}^{\alpha+2}y\right)(x)+\mu x^{\alpha+1}\left(D_{-}^{\alpha+1}y\right)(x)+\lambda x^{\alpha}\left(D_{-}^{\alpha}y\right)(x)=f(x)\quad(x>0),\tag{1.1}$$

with $\alpha > 0$ and complex $\delta, \mu, \lambda \in \mathbb{C}$ on a positive half-axis $\mathbb{R}_+ = (0; +\infty)$. Here $(D_-^{\alpha}y)(x)$ is the right-hand sided Liouville fractional derivative of order $\alpha > 0$, defined by [9, (5.8)]:

$$\left(D_{-}^{\alpha}y\right)(x)=\left(-\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\alpha)}\int_{x}^{\infty}\frac{y(t)dt}{(t-x)^{\alpha-n+1}}\quad(x>0;\,n=[\alpha]+1),\tag{1.2}$$

where $\Gamma(n-\alpha)$ is the Euler gamma function [2, Section 1.1], and $[\alpha]$ is the integer part of $\alpha > 0$.

When $\alpha = m$, $m \in \mathbb{N}$ is a natural, then in accordance with (1.2),

$$(D_{-}^{m}y)(x)=(-1)^{m}y^{(m)}(x)\quad(m\in\mathbb{N}),\tag{1.3}$$

and equation (1.1) is reduced to the linear ordinary differential equation of order $m+2$ $(m \in \mathbb{N})$:

$$\delta_1 x^{m+2}y^{(m+2)}(x)+\mu_1 x^{m+1}y^{(m+1)}(x)+\lambda_1 x^{m}y^{(m)}(x)=f(x)\quad(x>0),\tag{1.4}$$

$$\delta_1=(-1)^{m}\delta,\quad \mu_1=(-1)^{m+1}\mu,\quad \lambda_1=(-1)^{m}\lambda.$$

This equation is known as the Euler equation, and its solution is reduced to solution of the linear ordinary differential equation with constant coefficients by using the change $t = \log(x)$; for example, see [14, 5.2.27]. Therefore we call (1.1) Euler-type fractional diferential equation.

We solve equation (1.1) by applying the one-dimensional direct and inverse Mellin transforms $\mathcal{M}$ and $\mathcal{M}^{-1}$. The direct Mellin transform $\mathcal{M}\varphi$ of a function $\varphi(t)$ of real variable $t \in \mathbb{R}_+$ is defined by

$$\left(\mathcal{M}\varphi\right)(s)=\int_{0}^{\infty}t^{s-1}\varphi(t)dt\quad(s\in\mathbb{C}),\tag{1.5}$$

and the inverse Mellin transform $\mathcal{M}^{-1}g$ is given for $x \in \mathbb{R}_+$ by the formula

$$(\mathcal{M}^{-1}g)(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}g(s)ds\quad(c=\Re(s)).\tag{1.6}$$

One may find properties of these Mellin transforms in the books [1] and [10].

We use such an approach and the residue theory to establish explicit solutions of equation (1.1) in terms of special cases of the generalized Wright function ${}_p\Psi_q[z]$ and of the generalized hypergeometric function ${}_pF_q[z]$ as well as in terms of the Euler psi-function [2, Section 1.7]

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz} \ln \Gamma(z). \tag{1.7}$$

The generalized Wright function ${}_p\Psi_q[z]$ is defined for complex $z, a_i, b_j \in \mathbb{C}$ and real $\alpha_i, \beta_j \in \mathbb{R}$ $(\alpha_i, \beta_j \neq 0; i = 1, 2, ..., p; j = 1, 2, ..., q)$ by the series

$${}_p\Psi_q[z] \equiv {}_p\Psi_q\left[\begin{array}{c} (a_1, \alpha_1), ..., (a_p, \alpha_p) \\ (b_1, \beta_1), ..., (b_q, \beta_q) \end{array} \mid z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}. \tag{1.8}$$

This function was introduced by Wright [11] and is called his name; see [2, Section 4.1]. Wright [11]—[13] proved main terms of its asymptotic expansions at infinity under the condition

$$\Delta \equiv \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1. \tag{1.9}$$

The special case of function (1.8) in the form

$$J_\nu^\mu(z) \equiv {}_0\Psi_1\left[\begin{array}{c} ---- \\ (\nu + 1, \mu) \end{array} \mid z\right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\mu k + \nu + 1)} \frac{(-z)^k}{k!} \tag{1.10}$$

is known as the Besel-Maitland function, or the Wright generalized Bessel function; see [6, p. 352] and [7, (8.3)]. When $\nu = \beta - 1$, $\mu = \alpha$ and z is replaced by $-z$, the function $J_{\beta-1}^\alpha(-z)$ is denoted by $\varphi(\alpha, \beta; z)$:

$$\varphi(\alpha, \beta; z) \equiv J_{\beta-1}^\alpha(-z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \tag{1.11}$$

and such a function is known as the Wright function; see [3, Section 18.1].

Conditions for the existence of the generalized Wright function (1.8) together with its representations in terms of the Mellin-Barnes integrals and of the so-called H-functions were established in [4]. In particular, it was proved that ${}_p\Psi_q[z]$ is an entire function of $z \in \mathbb{C}$ provided that condition (1.9) is valid.

The generalized hypergeometric function ${}_pF_q[z]$ is defined for $z \in \mathbb{C}$ and $a_i, b_j \in \mathbb{C},\ b_j \neq 0$ $(i = 1, 2, ..., p;\ j = 1, ..., q)$ by the series [2, 4.1(1)]:

$$ {}_pF_q[z] \equiv {}_pF_q\left[\begin{array}{c} a_1, ..., a_p \\ b_1, ..., b_q \end{array} \mid z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}. \tag{1.12} $$

Here $(z)_k$ is the Pochhammer symbol defined for complex $z \in \mathbb{C}$ and non-negative integer $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ by

$$ (z)_0 = 1,\ (z)_k = z(z+1)\cdots(z+k-1) \quad (k \in \mathbb{N}). $$

It is known (see, for example, [2, Section 4.1]) that the series (1.12) converges absolutely for any $z \in \mathbb{C}$ if $p \leq q$, while if $p = q+1$, for $|z| < 1$ and $|z| = 1$,

$$ \Re\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) > 0. \tag{1.13} $$

Note that when $\alpha_i = \beta_j = 1$ $(i = 1, ..., p;\ j = 1, ..., q)$, the generalized Wright function (1.8) coincides, apart from a multiplicative factor, with the generalized hypergeometric function (1.12):

$$ \frac{1}{\prod_{i=1}^{p} \Gamma(a_i)}\ {}_p\Psi_q\left[\begin{array}{c} (a_1, 1), ..., (a_p, 1) \\ (b_1, 1), ..., (b_q, 1) \end{array} \mid z\right] = \frac{1}{\prod_{j=1}^{q} \Gamma(b_j)}\ {}_pF_q\left[\begin{array}{c} a_1, ..., a_p \\ b_1, ..., b_q \end{array} \mid z\right]. $$

Our investigations are based on the results from [5, Section 5.4.1], where a general approach based on the Mellin transforms (1.5) and (1.6) was develop to solution of the equation

$$ \sum_{k=0}^{m} B_k x^{\alpha+k} \left(D_-^{\alpha+k} y\right)(x) = f(x) \quad (x > 0;\ \alpha > 0) $$

with real $B_k \in \mathbb{R}$ $(k = 0, 1, ..., m)$. In particular, it was proved that the explicit solutions of the equation

$$ x^{\alpha+1} \left(D_-^{\alpha+1} y\right)(x) + \lambda x^{\alpha} \left(D_-^{\alpha} y\right)(x) = f(x) \quad (x > 0;\ \alpha > 0) \tag{1.14} $$

with real $\lambda \in \mathbb{R}$ are expressed in terms of the generalized Wright function ${}_1\Psi_2[z]$ and $\psi(z)$ [5, Section 5.4.3].

Note that Podlubny [8, Section 6.1] indicated that the Mellin transform (1.5) can be applied to obtaining explicit solution of the Cauchy-type problem for equation of the form (1.14) (with $\lambda = 1$ and $0 < \alpha < 1$) in which the right-sided fractional derivatives $D_{-}^{\alpha+1}y$ and $D_{-}^{\alpha}y$ are replaced by the left-hand sided ones $D_{0+}^{\alpha+1}y$ and $D_{0+}^{\alpha}y$; see in this connection [5, Sections 5.1 and 5.4.2].

The paper is organized as follows. In Section 2, explicit solution of equation (1.1) is established in terms of the direct and inverse Mellin transforms (1.5) and (1.6). In Section 3 the obtained explicit solutions are expressed via generalized Wright function (1.8), generalized hypergeometric series (1.12) and the Euler psi-function (1.7). Section 4 deals with solution of equation (1.1) in special cases. Solutions of ordinary differential equation (1.4) are presented in Section 5. Examples are given in Section 6.

2. Solution in terms of direct and inverse Mellin transforms

The Mellin transform method for solving equation (1.1) is based on the relation

$$\left(\mathcal{M}x^{\alpha+k}D_{-}^{\alpha+k}y\right)(s) = \frac{\Gamma(s+\alpha+k)}{\Gamma(s)}(\mathcal{M}y)(s), \tag{2.1}$$

being valid for suitable function $y(x)$; see [9, Section 7.3].

Applying the Mellin transform (1.5) to (1.1) and taking (2.1) into account, we have

$$\left[\delta\frac{\Gamma(s+\alpha+2)}{\Gamma(s)} + \lambda\frac{\Gamma(s+\alpha)}{\Gamma(s)} + \mu\frac{\Gamma(s+\alpha+1)}{\Gamma(s)}\right](\mathcal{M}y)(s) = (\mathcal{M}f)(s). \tag{2.2}$$

Using the relation

$$\Gamma(z+1) = z\Gamma(z) \quad (z \in \mathbb{C}) \tag{2.3}$$

for the Euler gamma-function $\Gamma(z)$ [2, 1.2(1)], we rewrite (2.2) as

$$P_{\alpha}(s)(\mathcal{M}y)(s) = (\mathcal{M}f)(s), \tag{2.4}$$

$$P_{\alpha}(s) = \frac{\Gamma(s+\alpha)}{\Gamma(s)}\left[\delta s^2 + (\delta+\mu+2\delta\alpha)s + \delta\alpha^2 + (\delta+\mu)\alpha + \lambda\right].$$

Applying the inverse Mellin transform (1.6) to (2.4) and using the equality $\mathcal{M}^{-1}\mathcal{M}f = f$, being valid for suitable f, we derive the solution of (1.1) in the form

$$y(x) = \left(\mathcal{M}^{-1}\left[\frac{\mathcal{M}f(s)}{P_\alpha(s)}\right]\right)(x)\,. \tag{2.5}$$

Now we introduce the function

$$G_{\alpha;\,\delta,\mu,\lambda}(x) = \left(\mathcal{M}^{-1}\left[\frac{1}{P_\alpha(s)}\right]\right)(x)$$

$$= \left(\mathcal{M}^{-1}\left[\frac{\Gamma(s)}{(\delta s^2 + (\delta+\mu+2\delta\alpha)s + \delta\alpha^2 + (\delta+\mu)\alpha + \lambda)\Gamma(s+\alpha)}\right]\right)(x)\,. \tag{2.6}$$

Let $\delta \neq 0$ and let s_1, s_2 be the roots of the equation

$$\delta s^2 + (\delta+\mu+2\delta\alpha)s + \delta\alpha^2 + (\delta+\mu)\alpha + \lambda = 0, \tag{2.7}$$

given by

$$s_{1,2} = \frac{-\delta-\mu-2\delta\alpha \pm \sqrt{D}}{2\delta}, \quad D = (\delta+\mu+2\delta\alpha)^2 - 4\delta[\delta\alpha^2 + (\delta+\mu)\alpha + \lambda]. \tag{2.8}$$

Then

$$\delta s^2 + (\delta+\mu+2\delta\alpha)s + \delta\alpha^2 + (\delta+\mu)\alpha + \lambda = \delta(s-s_1)(s-s_2)$$

and (2.6) is represented as

$$G_{\alpha;\,\delta,\mu,\lambda}(x) = \frac{1}{\delta}\left(\mathcal{M}^{-1}\left[\frac{\Gamma(s)}{\Gamma(s+\alpha)(s-s_1)(s-s_2)}\right]\right)(x)\,. \tag{2.9}$$

Applying the modified Mellin convolution property

$$\left(\mathcal{M}\int_0^\infty K\left(\frac{x}{t}\right)f(t)\frac{dt}{t}\right)(s) = (\mathcal{M}K)(s)(\mathcal{M}f)(s),$$

with $K(x) = G_{\alpha;\,\delta,\mu,\lambda}(x)$, we present solution (2.5) in the form

$$y(x) = \int_0^\infty G_{\alpha;\,\delta,\mu,\lambda}\left(\frac{x}{t}\right)f(t)\frac{dt}{t}. \tag{2.10}$$

Now we show that $G_{\alpha;\,\delta,\mu,\lambda}(x) = 0$ for $x > 1$. By (2.9) we have

$$\left(\mathcal{M}G_{\alpha;\,\delta,\mu,\lambda}\right)(s) = \frac{1}{\delta}\left(\mathcal{M}G_1\right)(s)\left(\mathcal{M}G_2\right)(s)\,,$$

where

$$(\mathcal{M}G_1)(s) = \frac{\Gamma(s)}{\Gamma(s+\alpha)}; \; (\mathcal{M}G_2)(s) = \frac{1}{(s-s_1)(s-s_2)}.$$

The direct application of the Mellin transform (1.5) leads to the following relations:

$$G_1(x) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)} & , \quad 0 < x < 1, \\ 0 & , \quad x > 1; \end{cases} \tag{2.11}$$

$$G_2(x) = \begin{cases} \frac{1}{s_1 - s_2}(x^{-s_1} - x^{-s_2}) & , \quad 0 < x < 1, \\ 0 & , \quad x > 1; \end{cases} \qquad (s_1 \neq s_2) \tag{2.12}$$

$$G_2(x) = \begin{cases} -x^{-s_1}\log(x) & , \quad 0 < x < 1, \\ 0 & , \quad x > 1. \end{cases} \qquad (s_1 = s_2). \tag{2.13}$$

Then, according to the Mellin convolution property

$$\left(\mathcal{M}\int_0^\infty K\left(\frac{x}{t}\right) f(t)\frac{dt}{t}\right)(s) = (\mathcal{M}K)(s)\,(\mathcal{M}f)(s),$$

with $K(x) = G_1(x)$ and $f(x) = G_2(x)$, we have

$$G_{\alpha;\,\delta,\mu,\lambda}(x) = \int_0^\infty G_1\left(\frac{x}{t}\right) G_2(t)\frac{dt}{t},$$

and it follows from (2.11) – (2.13) that $G_{\alpha;\,\delta,\mu,\lambda}(x) = 0$ for $x > 1$.

Hence (2.10) yields the solution of equation (1.1):

$$y(x) = \int_x^\infty G_{\alpha;\,\delta,\mu,\lambda}\left(\frac{x}{t}\right) f(t)\frac{dt}{t}, \tag{2.14}$$

or

$$y(x) = \int_1^\infty G_{\alpha;\,\delta,\mu,\lambda}\left(\frac{1}{t}\right) f(xt)\frac{dt}{t}, \tag{2.15}$$

where according to (1.6) and (2.9)

$$G_{\alpha;\,\delta,\mu,\lambda}(x) = \frac{1}{2\pi i\delta}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{\Gamma(s+\alpha)(s-s_1)(s-s_2)} x^{-s} ds \quad (c = \Re(s)). \tag{2.16}$$

By analogy with ordinary differential equations, $G_{\alpha;\,\delta,\mu,\lambda}(x)$ can be called the Mellin fractional analogue of the Green function.

It is directly verified that $y(x)$ in (2.14) is a particular solution of equation (1.1).

We evaluate the integral in (2.16) by using the residue theory. It is known (see, for example, [2, Section 1.1]) that $\Gamma(s)$ is analytic in $\mathbb{C}$ except simple poles $s=-n$ $(n\in\mathbb{N})$ with the residue $\frac{(-1)^n}{n!}$, and

$$\Gamma(s)\sim\frac{(-1)^n}{n!(s+n)}\quad (s\to -n,\ n\in\mathbb{N}_0). \tag{2.17}$$

Therefore the integrand in the right-hand side of (2.16)

$$g(s)=\frac{\Gamma(s)}{\Gamma(s+\alpha)(s-s_1)(s-s_2)}x^{-s} \tag{2.18}$$

for any fixed $x\neq 0$ is analytic function of s except simple poles $s=s_1,\ s=s_2$ and $s=-n$ $(n\in\mathbb{N}_0)$. Therefore $G_{\alpha;\,\delta,\mu,\lambda}(x)$ in (2.16) will have different values in the following 5 cases:

Case 1. $s_1\neq s_2\neq -k$ for any $k\in\mathbb{N}_0$.

Case 2. $s_1=s_2\neq -k$ for any $k\in\mathbb{N}_0$.

Case 3. There exists $n_0\in\mathbb{N}_0$, such that $s_1\neq s_2,\ s_1=-n_0$.

Case 4. There exists $n_0\in\mathbb{N}_0$, such that $s_1\neq s_2,\ s_2=-n_0$.

Case 5. There exists $n_0\in\mathbb{N}_0$, such that $s_1=s_2=-n_0$.

Further we show that $G_{\alpha;\,\delta,\mu,\lambda}(x)$ is expressed in terms of special cases of the generalized Wright function (1.8), of the generalized hypergeometric function (1.12) and of the Euler psi function (1.7).

We need preliminary assertion giving conditions for the existence of ${}_p\Psi_q[z]$ proved in [4, Theorem 2]; see also [5, Theorem 1.5].

LEMMA 1. *Let $a_i,b_j\in\mathbb{C}$ and $\alpha_i,\beta_j\in\mathbb{R}$ $(i=1,...,p,\ j=1,...,q)$, let Δ be given by (1.9) and let*

$$\delta=\prod_{i=1}^{p}|\alpha_i|^{-\alpha_i}\prod_{j=1}^{q}|\beta_j|^{\beta_j},\quad \mu=\sum_{j=1}^{q}b_j-\sum_{i=1}^{p}a_i+\frac{p-q}{2}. \tag{2.19}$$

(i) If $\Delta>-1$, then the series in (1.8) is absolutely convergent for all $z\in\mathbb{C}$.

(ii) If $\Delta=-1$, then the series in (1.8) is absolutely convergent for $|z|<\delta$ and $|z|=\delta,\ \Re(\mu)>\frac{1}{2}$.

3. Solutions in terms of special functions

The first result yields the explicit solution of equation (1.1) in Case 1.

THEOREM 1. *Let $\alpha > 0$ and $\delta, \lambda, \mu \in \mathbb{C}$ ($\delta \neq 0$) and let roots s_1, s_2 in (2.8) be such that $s_1 \neq s_2 \neq -k$ for any $k \in \mathbb{N}_0$. Then*

$$G_{\alpha;\,\delta,\mu,\lambda}(x) = \frac{1}{\delta}\,{}_2\Psi_3\left[\begin{matrix}(s_1,1),(s_2,1)\\(\alpha,-1),(s_1+1,1),(s_2+1,1)\end{matrix}\Big|\,-x\right]$$

$$+\frac{1}{\delta(s_1-s_2)}\left[\frac{\Gamma(s_1)x^{-s_1}}{\Gamma(s_1+\alpha)}-\frac{\Gamma(s_2)x^{-s_2}}{\Gamma(s_2+\alpha)}\right] \tag{3.1}$$

$$=\frac{1}{\delta s_1 s_2\Gamma(\alpha)}\,{}_3F_2\left[\begin{matrix}1-\alpha,s_1,s_2\\s_1+1,s_2+1\end{matrix}\Big|\,x\right]$$

$$+\frac{1}{\delta(s_1-s_2)}\left[\frac{\Gamma(s_1)}{\Gamma(s_1+\alpha)}x^{-s_1}-\frac{\Gamma(s_2)}{\Gamma(s_2+\alpha)}x^{-s_2}\right]. \tag{3.2}$$

The particular solution $y(x)$ of equation (1.1) is given by (2.14) and (2.15) provided that the integrals in the right-hand sides of (2.14) and (2.15) are convergent.

P r o o f. By the conditions of the theorem $s = s_1$, $s = s_2$ and $s = -k$ ($k \in \mathbb{N}_0$) are simple poles of the integrand (2.18) in (2.16) and they do not coincide. If we choose $c > \max[\Re(s_1), \Re(s_2), \alpha]$, then applying the usual technique for evaluating the Mellin-Barnes integral in (2.16) (for example, see [2, Section 2.1.3]), and taking (2.17) into account we have

$$G_{\alpha;\,\delta,\mu,\lambda}(x) = \frac{1}{\delta}\left[\sum_{k=0}^{\infty}\mathrm{res}_{s=-k}g(s)+\mathrm{res}_{s=s_1}g(s)+\mathrm{res}_{s=s_2}g(s)\right]$$

$$=\frac{1}{\delta}\sum_{k=0}^{\infty}\lim_{s\to-k}\frac{(s+k)\Gamma(s)x^{-s}}{\Gamma(s+\alpha)(s-s_1)(s-s_2)}+\frac{1}{\delta}\lim_{s\to s_1}\frac{(s-s_1)\Gamma(s)x^{-s}}{\Gamma(s+\alpha)(s-s_1)(s-s_2)}$$

$$+\frac{1}{\delta}\lim_{s\to s_2}\frac{(s-s_2)\Gamma(s)x^{-s}}{\Gamma(s+\alpha)(s-s_1)(s-s_2)}=\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!(-k-s_1)(-k-s_2)\Gamma(\alpha-k)}$$

$$+\frac{\Gamma(s_1)}{\delta(s_1-s_2)\Gamma(s_1+\alpha)}x^{-s_1}+\frac{\Gamma(s_2)}{\delta(s_2-s_1)\Gamma(s_2+\alpha)}x^{-s_2}. \tag{3.3}$$

Using the relation (2.3), we find

$$G_{\alpha;\,\delta,\mu,\lambda}(x)=\frac{1}{\delta}\sum_{k=0}^{\infty}\frac{\Gamma(s_1+k)\Gamma(s_2+k)(-x)^k}{\Gamma(\alpha-k)\Gamma(s_1+1+k)\Gamma(s_2+1+k)k!}$$

$$+\frac{1}{\delta}\left[\frac{\Gamma(s_1)}{(s_1-s_2)\Gamma(s_1+\alpha)}x^{-s_1}-\frac{\Gamma(s_2)}{(s_1-s_2)\Gamma(s_2+\alpha)}x^{-s_2}\right]. \tag{3.4}$$

This yields (3.1) in accordance with (1.8). The constants in (1.9) and (2.19) take the forms

$$\Delta=-1,\ \delta=1,\ \mu=\alpha+\frac{3}{2}. \tag{3.5}$$

Therefore the function ${}_2\Psi_3[-x]$ exists by Lemma 1(ii).

Using the formulas

$$\frac{1}{\Gamma(\alpha-k)}\equiv\frac{(-1)^k(1-\alpha)_k}{\Gamma(\alpha)}\quad(\alpha\in\mathbb{C},\ k\in\mathbb{N}_0), \tag{3.6}$$

$$\Gamma(z+k)=\Gamma(z)(z)_k\quad(z\in\mathbb{C},\ k\in\mathbb{N}_0), \tag{3.7}$$

from (3.4) we deduce (3.2). By (1.13), the function ${}_3F_2$ in (3.2) exists. Thus the theorem is proved. ■

Next statement yields explicit solution $y(x)$ of equation (1.1) in case 2.

THEOREM 2. *Let $\alpha>0$ and $\delta,\lambda,\mu\in\mathbb{C}$ ($\delta\neq 0$) and let roots s_1, s_2 in (2.8) be such that $s_1=s_2\neq -k$ for any $k\in\mathbb{N}_0$. Then*

$$G_{\alpha;\,\delta,\mu,\lambda}(x)=\frac{1}{\delta}\,{}_2\Psi_3\left[\begin{matrix}(s_1,1),(s_1,1)\\(\alpha,-1),(s_1+1,1),(s_1+1,1)\end{matrix}\Big|\,-x\right]$$

$$+\frac{\Gamma(s_1)x^{-s_1}}{\delta\Gamma(s_1+\alpha)}[\psi(s_1)-\psi(s_1+\alpha)-\log(x)] \tag{3.8}$$

$$=\frac{1}{\delta {s_1}^2\Gamma(\alpha)}\,{}_3F_2\left[\begin{matrix}1-\alpha,s_1,s_1\\s_1+1,s_1+1\end{matrix}\Big|\,x\right]$$

$$+\frac{\Gamma(s_1)}{\delta\Gamma(s_1+\alpha)}x^{-s_1}[\psi(s_1)-\psi(s_1+\alpha)-\log(x)]. \tag{3.9}$$

The particular solution $y(x)$ of equation (1.1) is given by (2.14) and (2.15) provided that the integrals in the right-hand sides of (2.14) and (2.15) are convergent.

P r o o f. By conditions of the theorem, the integrand (2.18) in (2.16) has a pole of second order at $s = s_1 = s_2$ and simple poles at $s = -k$ for $k \in \mathbb{N}_0$. If we choose $c > \max[\Re(s_1), \alpha]$, then similarly to (3.3) and (3.4) we have

$$\begin{aligned} G_{\alpha;\,\delta,\mu,\lambda}(x) &= \frac{1}{\delta}\left[\sum_{k=0}^{\infty} \mathrm{res}_{s=-k}\, g(s) + \mathrm{res}_{s=s_1}\, g(s)\right] \\ &= \frac{1}{\delta}\left[\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!(-k-s_1)^2\Gamma(\alpha-k)} + \lim_{s\to s_1}\left(\frac{\Gamma(s)x^{-s}}{\Gamma(s+\alpha)}\right)'\right] \\ &= \frac{1}{\delta}\sum_{k=0}^{\infty} \frac{\Gamma^2(s_1+k)(-x)^k}{k!\Gamma^2(s_1+1+k)\Gamma(\alpha-k)} + \frac{\Gamma'(s_1)}{\delta\Gamma(s_1+\alpha)}x^{-s_1} \\ &\quad - \frac{\Gamma(s_1)\Gamma'(s_1+\alpha)}{\delta\Gamma^2(s_1+\alpha)}x^{-s_1} - \frac{\Gamma(s_1)}{\delta\Gamma(s_1+\alpha)}x^{-s_1}\log(x). \end{aligned} \tag{3.10}$$

According to (1.7) and (1.8), this yields (3.8). (3.9) is deduced from (3.8). By the proof of Theorem 1, ${}_2\Psi_3[-x]$ in (3.8) and ${}_3F_2$ in (3.9) exist. This completes the proof of Theorem 2. ■

Let

$$\gamma = -\psi(1); \quad \sum_{k=0}^{n} a_k = 0 \quad (n = -1, -2, ...). \tag{3.11}$$

Now we obtain the particular solution $y(x)$ of equation (1.1) in case 3.

THEOREM 3. *Let $\alpha > 0$ and $\delta, \lambda, \mu \in \mathbb{C}$ ($\delta \neq 0$) and let roots s_1 and s_2 in (2.8) be such that there exists $n_0 \in \mathbb{N}_0$ such that $s_1 \neq s_2$, $s_1 = -n_0$. Then*

$$\begin{aligned} G_{\alpha;\,\delta,\mu,\lambda}(x) &= \frac{1}{\delta}\sum_{k=0}^{n_0-1} \frac{(-1)^k x^k}{k!(k-n_0)(k+s_2)\Gamma(\alpha-k)} + \frac{\Gamma(s_2)}{\delta\Gamma(s_2+\alpha)}\frac{x^{-s_2}}{(s_2+n_0)} \\ &+ \frac{(-1)^{n_0+1}x^{n_0+1}}{\delta}\,{}_3\Psi_4\left[\begin{matrix}(1,1),(1,1),(n_0+1+s_2,1) \\ (2,1),(n_0+2,1),(n_0+2+s_2,1),(\alpha-n_0-1,-1)\end{matrix}\,\middle|\, -x\right] \\ &+ \frac{(-1)^{n_0}x^{n_0}}{n_0!\delta(n_0+s_2)\Gamma(\alpha-n_0)}\left[\gamma + \psi(\alpha-n_0) - \frac{1}{n_0+s_2} + \sum_{j=0}^{n_0-1}\frac{1}{j-n_0} + \log(x)\right] \end{aligned} \tag{3.12}$$

$$= \frac{1}{\delta}\sum_{k=0}^{n_0-1} \frac{(-1)^k x^k}{k!(k-n_0)(k+s_2)\Gamma(\alpha-k)} + \frac{\Gamma(s_2)}{\delta\Gamma(s_2+\alpha)}\frac{x^{-s_2}}{(s_2+n_0)}$$

$$+\frac{(-1)^{n_0+1}x^{n_0+1}}{\delta\Gamma(n_0+2)\Gamma(\alpha-n_0-1)(s_2+n_0+1)}\,{}_4F_3\left[\begin{array}{c}1,1,1+n_0+s_2,2+n_0-\alpha\\ 2,n_0+2,2+n_0+s_2\end{array}\Big|\,x\right]$$

$$+\frac{(-1)^{n_0}x^{n_0}}{n_0!\delta(n_0+s_2)\Gamma(\alpha-n_0)}\left[\gamma+\psi(\alpha-n_0)-\frac{1}{n_0+s_2}+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}+\log(x)\right]. \tag{3.13}$$

The particular solution $y(x)$ of equation (1.1) is given by (2.14) and (2.15) provided that the integrals in the right-hand sides of (2.14) and (2.15) are convergent.

P r o o f. By conditions of the theorem, the integrand (2.18) in (2.16) has a pole of second order at $s = s_1 = -n_0$ and simple poles at $s = -k$ ($k \in \mathbb{N}_0$, $k \neq n_0$) and if we choose $c > \max[-n_0, \Re(s_2)]$, then similarly to (3.3) and (3.10) we have

$$G_{\alpha;\,\delta,\mu,\lambda}(x) = \frac{1}{\delta}\left[\sum_{k=0,\,k\neq n_0}^{\infty}\mathrm{res}_{s=-k}g(s)+\mathrm{res}_{s=s_1=-n_0}g(s)+\mathrm{res}_{s=s_2}g(s)\right]$$

$$=\frac{1}{\delta}\sum_{k=0,\,k\neq n_0}^{\infty}\frac{(-1)^k x^k}{k!(k-n_0)(k+s_2)\Gamma(\alpha-k)}+\frac{1}{\delta}\lim_{s\to s_1}\left(\frac{(s-s_1)^2\Gamma(s)x^{-s}}{(s-s_1)(s-s_2)\Gamma(s+\alpha)}\right)'$$

$$+\frac{\Gamma(s_2)x^{-s_2}}{\delta\Gamma(s_2+\alpha)(s_2+n_0)}. \tag{3.14}$$

Evaluate the second term in the right-hand side of (3.14):

$$\lim_{s\to s_1=-n_0}\left(\frac{(s-s_1)^2\Gamma(s)x^{-s}}{(s-s_1)(s-s_2)\Gamma(s+\alpha)}\right)'$$

$$=\lim_{s\to -n_0}\left(\frac{\Gamma(s+n_0+1)x^{-s}}{(s-s_2)s(s+1)...(s+n_0-1)\Gamma(s+\alpha)}\right)'$$

$$=\lim_{s\to -n_0}\frac{\Gamma'(s+n_0+1)x^{-s}}{(s-s_2)s(s+1)...(s+n_0-1)\Gamma(s+\alpha)}$$

$$-\lim_{s\to -n_0}\frac{\Gamma(s+n_0+1)x^{-s}\log(x)}{(s-s_2)s(s+1)...(s+n_0-1)\Gamma(s+\alpha)}$$

$$-\lim_{s\to -n_0}\frac{\Gamma(s+n_0+1)\Gamma'(s+\alpha)x^{-s}}{(s-s_2)s(s+1)...(s+n_0-1)\Gamma^2(s+\alpha)}$$

$$
-\lim_{s\to -n_0} \frac{\Gamma(s+n_0+1)x^{-s}}{(s-s_2)^2 s(s+1)...(s+n_0-1)\Gamma(s+\alpha)}
$$

$$
-\lim_{s\to -n_0} \frac{\Gamma(s+n_0+1)x^{-s}\sum_{j=0}^{n_0-1}\frac{1}{s+j}}{(s-s_2)s(s+1)...(s+n_0-1)\Gamma(s+\alpha)}
$$

$$
= \frac{(-1)^{n_0}x^{n_0}}{(n_0+s_2)\cdot n_0!\Gamma(\alpha-n_0)}\left[\gamma+\psi(\alpha-n_0)-\frac{1}{n_0+s_2}+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}+\log(x)\right].
$$

Substituting this result in (3.14) and taking (3.11) into account, we find

$$
G_{\alpha;\,\delta,\mu,\lambda}(x) = \frac{1}{\delta}\sum_{k=0,k\neq n_0}^{\infty}\frac{(-1)^k x^k}{k!(k-n_0)(k+s_2)\Gamma(\alpha-k)} + \frac{\Gamma(s_2)x^{-s_2}}{\delta\Gamma(s_2+\alpha)(s_2+n_0)}
$$

$$
+\frac{(-1)^{n_0}x^{n_0}}{\delta(n_0+s_2)\cdot n_0!\Gamma(\alpha-n_0)}\left[\gamma+\psi(\alpha-n_0)-\frac{1}{n_0+s_2}+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}+\log(x)\right]. \tag{3.15}
$$

For the first sum in the right-hand side of (3.15) we have

$$
\sum_{k=0,k\neq n_0}^{\infty}\frac{(-1)^k x^k}{k!(k-n_0)(k+s_2)\Gamma(\alpha-k)} = \sum_{k=0}^{n_0-1}\frac{(-1)^k x^k}{k!(k-n_0)(k+s_2)\Gamma(\alpha-k)}
$$

$$
+\sum_{k=n_0+1}^{\infty}\frac{(-1)^k x^k}{k!(k-n_0)(k+s_2)\Gamma(\alpha-k)} = \sum_{k=0}^{n_0-1}\frac{(-1)^k x^k}{k!(k-n_0)(k+s_2)\Gamma(\alpha-k)}
$$

$$
+\sum_{j=0}^{\infty}\frac{(-1)^{n_0+j+1}x^{n_0+j+1}}{(n_0+j+1)!(1+j)(n_0+j+1+s_2)\Gamma(\alpha-n_0-j-1)} = \sum_{k=0}^{n_0-1}\frac{(-1)^k x^k}{k!(k-n_0)(k+s_2)\Gamma(\alpha-k)}
$$

$$
+(-1)^{n_0+1}x^{n_0+1}\sum_{j=0}^{\infty}\frac{\Gamma(1+j)\Gamma(1+j)\Gamma(n_0+j+1+s_2)(-x)^j}{j!\Gamma(2+j)\Gamma(n_0+j+2)\Gamma(n_0+j+s_2+2)\Gamma(\alpha-n_0-1-j)}. \tag{3.16}
$$

Substituting this expression into (3.15), we obtain

$$
G_{\alpha;\,\delta,\mu,\lambda}(x) = \frac{1}{\delta}\sum_{k=0,k\neq n_0}^{\infty}\frac{(-1)^k x^k}{k!(k-n_0)(k+s_2)\Gamma(\alpha-k)} + \frac{\Gamma(s_2)x^{-s_2}}{\delta\Gamma(s_2+\alpha)(s_2+n_0)}
$$

$$+\frac{(-1)^{n_0+1}x^{n_0+1}}{\delta}\sum_{j=0}^{\infty}\frac{\Gamma(1+j)\Gamma(1+j)\Gamma(n_0+j+1+s_2)(-x)^j}{j!\Gamma(2+j)\Gamma(n_0+j+2)\Gamma(n_0+j+s_2+2)\Gamma(\alpha-n_0-1-j)}$$

$$+\frac{(-1)^{n_0}x^{n_0}}{\delta(n_0+s_2)\cdot n_0!\Gamma(\alpha-n_0)}\left[\gamma+\psi(\alpha-n_0)-\frac{1}{n_0+s_2}+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}+\log(x)\right]. \tag{3.17}$$

This yields (3.12) in accordance with (1.8). The constants in (1.9) and (2.19) take the forms (3.5), and by Lemma 1(ii) the function ${}_3\Psi_4[-x]$ in (3.12) exists.

The second formula (3.13) follow from (3.17) if we take (3.6) and (3.7) into account. By (1.13), the function ${}_4F_3$ in (3.13) exists. Thus Theorem 3 is proved. ∎

The following assertion, presented case 4, is proved similarly to Theorem 3 by rearranging s_1 and s_2.

Theorem 4. *Let $\alpha>0$ and $\delta,\lambda,\mu\in\mathbb{C}$ ($\delta\neq 0$) and let roots s_1, s_2 in (2.8) be such that there exists $n_0\in\mathbb{N}_0$ such that $s_1\neq s_2$, $s_2=-n_0$. Then*

$$G_{\alpha;\,\delta,\mu,\lambda}(x)=\frac{1}{\delta}\sum_{k=0}^{n_0-1}\frac{(-1)^k x^k}{k!(k-n_0)(k+s_1)\Gamma(\alpha-k)}+\frac{\Gamma(s_1)}{\delta\Gamma(s_1+\alpha)}\frac{x^{-s_1}}{(s_1+n_0)}$$

$$+\frac{(-1)^{n_0+1}x^{n_0+1}}{\delta}\,{}_3\Psi_4\left[\begin{matrix}(1,1),(1,1),(n_0+1+s_1,1)\\(2,1),(n_0+2,1),(n_0+2+s_1,1),(\alpha-n_0-1,-1)\end{matrix}\Big|\,-x\right]$$

$$+\frac{(-1)^{n_0}x^{n_0}}{n_0!\delta(n_0+s_1)\Gamma(\alpha-n_0)}\left[\gamma+\psi(\alpha-n_0)-\frac{1}{n_0+s_1}+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}+\log(x)\right] \tag{3.18}$$

$$=\frac{1}{\delta}\sum_{k=0}^{n_0-1}\frac{(-1)^k x^k}{k!(k-n_0)(k+s_1)\Gamma(\alpha-k)}+\frac{\Gamma(s_1)}{\delta\Gamma(s_1+\alpha)}\frac{x^{-s_1}}{(s_1+n_0)}$$

$$+\frac{(-1)^{n_0+1}x^{n_0+1}}{\delta\Gamma(n_0+2)\Gamma(\alpha-n_0-1)(s_1+n_0+1)}\,{}_4F_3\left[\begin{matrix}1,1,1+n_0+s_1,2+n_0-\alpha\\2,n_0+2,2+n_0+s_1\end{matrix}\Big|\,x\right]$$

$$+\frac{(-1)^{n_0}x^{n_0}}{n_0!\delta(n_0+s_1)\Gamma(\alpha-n_0)}\left[\gamma+\psi(\alpha-n_0)-\frac{1}{n_0+s_1}+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}+\log(x)\right]. \tag{3.19}$$

The particular solution $y(x)$ of equation (1.1) is given by (2.14) and (2.15) provided that the integrals in the right-hand sides of (2.14) and (2.15) are convergent.

The last result yields the particular solution $y(x)$ of equation (1.1) in case 5.

THEOREM 5. *Let $\alpha > 0$ and $\delta, \lambda, \mu \in \mathbb{C}$ ($\delta \neq 0$) and let roots s_1, s_2 in (2.8) be such that there exists $n_0 \in \mathbb{N}_0$ such that $s_1 = s_2 = -n_0$. Then*

$$G_{\alpha;\,\delta,\mu,\lambda}(x)=\frac{1}{\delta}\sum_{k=0}^{n_0-1}\frac{(-1)^k x^k}{k!(n_0-k)^2\Gamma(\alpha-k)}$$

$$+\frac{(-1)^{n_0+1}x^{n_0+1}}{\delta}\,{}_3\Psi_4\left[\begin{matrix}(1,1),(1,1),(1,1)\\(2,1),(2,1),(n_0+2,1),(\alpha-n_0-1,-1)\end{matrix}\,\middle|\,-x\right]$$

$$+\frac{(-1)^{n_0}}{2\delta n_0!\Gamma(\alpha-n_0)}$$

$$\times\left(\left[\gamma+\psi(\alpha-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}\right]^2+\psi'(1)-\psi'(\alpha-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{(j-n_0)^2}\right)x^{n_0}$$

$$+\frac{(-1)^{n_0}}{\delta n_0!\Gamma(\alpha-n_0)}\left[\gamma+\psi(\alpha-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}\right]x^{n_0}\log(x)$$

$$+\frac{(-1)^{n_0}}{2\delta n_0!\Gamma(\alpha-n_0)}x^{n_0}\log^2(x) \tag{3.20}$$

$$=\frac{1}{\delta}\sum_{k=0}^{n_0-1}\frac{(-1)^k x^k}{k!(n_0-k)^2\Gamma(\alpha-k)}+\frac{(-1)^{n_0+1}x^{n_0+1}}{\delta\Gamma(n_0+2)\Gamma(\alpha-n_0-1)}\,{}_4F_3\left[\begin{matrix}1,1,1,2-\alpha+n_0\\2,2,n_0+2\end{matrix}\,\middle|\,x\right]$$

$$+\frac{(-1)^{n_0}}{2\delta n_0!\Gamma(\alpha-n_0)}$$

$$\times\left(\left[\gamma+\psi(\alpha-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}\right]^2+\psi'(1)-\psi'(\alpha-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{(j-n_0)^2}\right)x^{n_0}$$

$$+\frac{(-1)^{n_0}}{\delta n_0!\Gamma(\alpha-n_0)}\left[\gamma+\psi(\alpha-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}\right]x^{n_0}\log(x)$$

$$+\frac{(-1)^{n_0}}{2\delta n_0!\Gamma(\alpha-n_0)}x^{n_0}\log^2(x). \tag{3.21}$$

The particular solution $y(x)$ of equation (1.1) is given by (2.14) and (2.15) provided that the integrals in the right-hand sides of (2.14) and (2.15) are convergent.

P r o o f. By conditions of the theorem, the integrand (2.18) in (2.16) has a pole of third order at $s=s_1=s_2=-n_0$ and simple poles at $s=-k$ ($k\in\mathbb{N}_0$, $k\neq n_0$). If we choose $c>\alpha$, then similarly to (3.3), (3.11) and (3.14) we have

$$G_{\alpha;\,\delta,\mu,\lambda}(x)=\frac{1}{\delta}\left[\sum_{k=0,k\neq n_0}^{\infty}\mathrm{res}_{s=-k}g(s)+\mathrm{res}_{s=-n_0}g(s)\right]$$

$$=\frac{1}{\delta}\sum_{k=0,k\neq n_0}^{\infty}\frac{(-1)^k x^k}{k!(n_0-k)^2\Gamma(\alpha-k)}+\frac{1}{2\delta}\lim_{s\to -n_0}\left[(s-s_1)^3\frac{\Gamma(s)}{(s-s_1)^2\Gamma(s+\alpha)}x^{-s}\right]''$$

$$=\frac{1}{\delta}\sum_{k=0,k\neq n_0}^{\infty}\frac{(-1)^k x^k}{k!(n_0-k)^2\Gamma(\alpha-k)}+\frac{1}{2\delta}\left[\lim_{s\to -n_0}\frac{\Gamma(s+n_0+1)x^{-s}}{s(s+1)...(s+n_0-1)\Gamma(s+\alpha)}\right]''. \tag{3.22}$$

The first term in the right-hand side is divided by two, as it was done in the proof of Theorem 3; see (3.16). For the second term in (3.22) the direct evaluation yields:

$$\lim_{s\to -n_0}\left[\frac{\Gamma(s+n_0+1)x^{-s}}{s(s+1)...(s+n_0-1)\Gamma(s+\alpha)}\right]''$$

$$=\frac{(-1)^{n_0}}{n_0!\Gamma(\alpha-n_0)}$$

$$\times\left(\left[\gamma+\psi(\alpha-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}\right]^2+\psi'(1)-\psi'(\alpha-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{(j-n_0)^2}\right)x^{n_0}$$

$$+\frac{2(-1)^{n_0}}{n_0!\Gamma(\alpha-n_0)}\left[\gamma+\psi(\alpha-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}\right]x^{n_0}\log(x)$$

$$+\frac{(-1)^{n_0}}{n_0!\Gamma(\alpha-n_0)}x^{n_0}\log^2(x). \tag{3.23}$$

Substituting (3.23) into (3.22) and taking (3.16) and (1.8) into account, we obtain (3.20). The constants in (1.9) and (2.19) are given by (3.5), and thus the function ${}_3\Psi_4[-x]$ in (3.20) exists. The second relation (3.21) follows from the first (3.20) if we take (3.6) and (3.7) into account. By (1.13), the function ${}_4F_3$ in (3.21) exists. This completes the proof of Theorem 5. ∎

Remark 1. The generalized hypergeometric function ${}_3F_2[x]$ in (3.2) can be expressed via the Gauss hypergeometric functions ${}_2F_1[x]$: for $s_1 \neq s_2$

$$ {}_3F_2\left[\begin{matrix}1-\alpha, s_1, s_2\\ s_1+1, s_2+1\end{matrix}\Big|\, x\right] = \frac{s_2}{s_2-s_1}\,{}_2F_1\left[\begin{matrix}1-\alpha, s_1\\ s_1+1\end{matrix}\Big|\, x\right] - \frac{s_1}{s_2-s_1}\,{}_2F_1\left[\begin{matrix}1-\alpha, s_2\\ s_2+1\end{matrix}\Big|\, x\right]. $$

4. Special cases

First consider equation (1.1) with $\delta = 0,\ \mu \neq 0,\ \lambda \neq 0:$

$$ \mu x^{\alpha+1}\left(D_-^{\alpha+1}y\right)(x) + \lambda x^{\alpha}\left(D_-^{\alpha}y\right)(x) = f(x) \quad (x > 0). \tag{4.1} $$

In this case (2.16) and (2.7)–(2.8) take the forms

$$ G_{\alpha;\,\mu,\lambda}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)x^{-s}}{[\mu s + \mu\alpha + \lambda]\Gamma(s+\alpha)}ds \quad (c = \Re(s)), \tag{4.2} $$

$$ \mu s + \mu\alpha + \lambda = 0, \tag{4.3} $$

$$ s_1 = -\alpha - \frac{\lambda}{\mu}, \tag{4.4} $$

and then (2.14) and (2.15) transform to:

$$ y(x) = \int_x^\infty G_{\alpha;\,\mu,\lambda}\left(\frac{x}{t}\right) f(t)\frac{dt}{t}, \tag{4.5} $$

$$ y(x) = \int_1^\infty G_{\alpha;\,\mu,\lambda}\left(\frac{1}{t}\right) f(xt)\frac{dt}{t}. \tag{4.6} $$

It is directly verified that $y(x)$ in (4.5) is a particular solution of equation (4.1). It have different forms in two cases:

(i) $\lambda \neq \mu(n-\alpha)$ for any $n \in \mathbb{N}_0$;

(ii) there exists $n_0 \in \mathbb{N}_0$ such that $\lambda = \mu(n_0 - \alpha)$.

Using (4.2)–(4.6) and taking the same arguments as in proofs of Theorems 1-5 in Section 3, we deduce the following results.

Theorem 6. *Let $\alpha > 0$ and $\mu, \lambda \in \mathbb{C}$ $(\mu \neq 0, \lambda \neq 0)$ be such that $\lambda \neq \mu(n-\alpha)$ for any $n \in \mathbb{N}_0$. Then*

$$G_{\alpha;\,\mu,\lambda}(x) = \frac{\Gamma(-\alpha-\frac{\lambda}{\mu})}{\mu\Gamma(-\frac{\lambda}{\mu})} x^{\alpha+\frac{\lambda}{\mu}} - {}_1\Psi_2\left[\begin{matrix} (-\mu\alpha-\lambda,\mu) \\ (\alpha,-1),(1-\mu\alpha-\lambda,\mu) \end{matrix}\Big|\, -x\right], \quad (4.7)$$

and the particular solution $y(x)$ of equation (4.1) is given by (4.5) and (4.6) provided that the integrals in the right-hand sides of (4.5) and (4.6) are convergent.

Theorem 7. *Let $\alpha > 0$ and $\mu, \lambda \in \mathbb{C}$ $(\mu \neq 0, \lambda \neq 0)$ such that there exists $n_0 \in \mathbb{N}_0$ such that $\lambda = \mu(n_0-\alpha)$. Then*

$$G_{\alpha;\,\mu,\lambda}(x) = \frac{1}{\mu}\sum_{k=0}^{n_0-1} \frac{(-1)^k x^k}{k!(n_0-k)\Gamma(\alpha-k)}$$

$$+\frac{(-1)^{n_0}x^{n_0+1}}{\mu}\,{}_2\Psi_3\left[\begin{matrix} (1,1), \quad (1,1) \\ (n_0+2,1),(2,1),(\alpha-n_0-1,-1) \end{matrix}\Big|\, -x\right]$$

$$-\frac{(-1)^{n_0}x^{n_0}}{n_0!\mu\Gamma(\alpha-n_0)}\left[\gamma+\psi(\alpha-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}+\log(x)\right] \quad (4.8)$$

$$= \frac{1}{\mu}\sum_{k=0}^{n_0-1} \frac{(-1)^k x^k}{k!(n_0-k)\Gamma(\alpha-k)} + \frac{(-1)^{n_0}x^{n_0+1}}{\mu\Gamma(n_0+2)\Gamma(\alpha-n_0-1)}\,{}_3F_2\left[\begin{matrix} 1,1,2-\alpha+n_0 \\ n_0+2,2 \end{matrix}\Big|\, x\right]$$

$$-\frac{(-1)^{n_0}x^{n_0}}{n_0!\mu\Gamma(\alpha-n_0)}\left[\gamma+\psi(\alpha-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}+\log(x)\right]. \quad (4.9)$$

The particular solution $y(x)$ of equation (4.1) is given by (4.5) and (4.6) provided that the integrals in the right-hand sides of (4.5) and (4.6) are convergent.

Remark 2. For $\lambda = 1$ results of Theorems 6 and 7 were proved in [5, Theorems 5.20 and 5.21].

Next consider equation (1.1) with $\lambda = 0$, $\delta \neq 0$, $\mu \neq 0$:

$$\delta x^{\alpha+2}\left(D_-^{\alpha+2}y\right)(x) + \mu x^{\alpha+1}\left(D_-^{\alpha+1}y\right)(x) = f(x) \quad (x>0). \quad (4.10)$$

Here (2.16) and (2.7)–(2.8) take the forms

$$G_{\alpha;\,\delta,\mu}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)x^{-s}}{[\delta s+\mu+\delta+\alpha\delta]\Gamma(s+\alpha+1)}ds \quad (c=\Re(s)), \tag{4.11}$$

$$\delta s+\mu+\delta+\alpha\delta = 0, \tag{4.12}$$

$$s_1 = -1-\alpha-\frac{\mu}{\delta}, \tag{4.13}$$

while (2.14) and (2.15) transform to

$$y(x) = \int_x^\infty G_{\alpha;\,\delta,\mu}\left(\frac{x}{t}\right) f(t)\frac{dt}{t}, \tag{4.14}$$

$$y(x) = \int_1^\infty G_{\alpha;\,\delta,\mu}\left(\frac{1}{t}\right) f(xt)\frac{dt}{t}. \tag{4.15}$$

It is directly verified that $y(x)$ in (4.14) is a particular solution of equation (4.10). It have different forms in two cases:

(i) $\mu \neq \delta(n-1-\alpha)$ for any $n \in \mathbb{N}_0$;

(ii) there exists $n_0 \in \mathbb{N}_0$ such that $\mu = \delta(n_0-1-\alpha)$.

Using (4.8)–(4.12) and taking the same arguments as in the proofs of Theorems 1-5, we deduce the following assertion.

THEOREM 8. *Let $\alpha > 0$ and $\lambda, \mu \in \mathbb{C}$ $(\lambda \neq 0, \mu \neq 0)$ be such that $\mu \neq \delta(n-1-\alpha)$ for any $n \in \mathbb{N}_0$. Then*

$$G_{\alpha;\,\delta,\mu}(x) = \frac{\Gamma(-1-\alpha-\frac{\mu}{\delta})}{\delta\Gamma(-\frac{\mu}{\delta})}x^{1+\alpha+\frac{\mu}{\delta}} - {}_1\Psi_2\left[\begin{matrix}(-\delta-\alpha\delta-\mu,\delta)\\(\alpha+1,-1),(-\delta-\alpha\delta-\mu+1,\delta)\end{matrix}\Big|\, -x\right], \tag{4.16}$$

and the particular solution $y(x)$ of equation (4.10) is given by (4.14) and (4.15) provided that the integrals in the right-hand sides of (4.14) and (4.15) are convergent.

THEOREM 9. *Let $\alpha > 0$ and $\lambda, \mu \in \mathbb{C}$ $(\lambda \neq 0, \mu \neq 0)$ be such that there exists $n_0 \in \mathbb{N}_0$ such that $\mu = \delta(n_0-1-\alpha)$. Then*

$$G_{\alpha;\,\delta,\mu}(x) = \frac{1}{\delta}\sum_{k=0}^{n_0-1}\frac{(-1)^k x^k}{k!(n_0-k)\Gamma(\alpha+1-k)}$$

$$+\frac{(-1)^{n_0}x^{n_0+1}}{\delta}\,{}_2\Psi_3\left[\begin{matrix}(1,1),\quad (1,1)\\(n_0+2,1),\ (2,1),\ (\alpha-n_0,-1)\end{matrix}\Big|\, -x\right]$$

$$-\frac{(-1)^{n_0}x^{n_0}}{n_0!\delta\Gamma(\alpha+1-n_0)}\left[\gamma+\psi(\alpha+1-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}+\log(x)\right] \tag{4.17}$$

$$=\frac{1}{\delta}\sum_{k=0}^{n_0-1}\frac{(-1)^k x^k}{k!(n_0-k)\Gamma(\alpha+1-k)}+\frac{(-1)^{n_0}x^{n_0+1}}{\delta\Gamma(n_0+2)\Gamma(\alpha-n_0)}\;{}_3F_2\left[\begin{array}{c}1,1,1+n_0-\alpha\\ n_0+2,2\end{array}\Big|\, x\right]$$

$$-\frac{(-1)^{n_0}x^{n_0}}{n_0!\delta\Gamma(\alpha+1-n_0)}\left[\gamma+\psi(\alpha+1-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}+\log(x)\right]. \tag{4.18}$$

The particular solution $y(x)$ of equation (4.10) is given by (4.14) and (4.15) provided that the integrals in the right-hand sides of (4.14) and (4.15) are convergent.

5. Ordinary differential equations

If $\alpha = m \in \mathbb{N}$, then in accordance with (1.3), formulas (2.14) and (2.15) give explicit solutions of the ordinary differential equation (1.4) in the form

$$y(x)=\int_x^{\infty} G_{m;\,\delta,\mu,\lambda}\left(\frac{x}{t}\right)f(t)\frac{dt}{t}\quad (m\in\mathbb{N}), \tag{5.1}$$

or

$$y(x)=\int_1^{\infty} G_{m;\,\delta,\mu,\lambda}\left(\frac{1}{t}\right)f(xt)\frac{dt}{t}\quad (m\in\mathbb{N}), \tag{5.2}$$

where according to (2.16) and (2.8)

$$G_{m;\,\delta,\mu,\lambda}(x)=\frac{(-1)^m}{2\pi i\delta}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)x^{-s}}{\Gamma(s+m)(s-s_1)(s-s_2)}ds\quad (c=\Re(s),\quad m\in\mathbb{N}), \tag{5.3}$$

$$s_{1,2}=-\frac{2m+1}{2}+\frac{\mu\pm(-1)^m\sqrt{D}}{2\delta},\ D=(\delta+2\delta m-\mu)^2-4\delta[\delta m^2+(\delta-\mu)m+\lambda]. \tag{5.4}$$

By (2.17), for $m\in\mathbb{N}$ and $k\in\mathbb{N}_0$,

$$\frac{1}{\Gamma(m-k)}=\frac{1}{(m-k-1)!}\ (k=0,1,...,m-1);\ \frac{1}{\Gamma(m-k)}=0\ \ (k=m,m+1,...). \tag{5.5}$$

Using these relations and taking (1.8) into account, from Theorems 1-5 we deduce the corresponding results for the ordinary differential equation (1.4). The first two results follow from Theorems 1 and 2.

THEOREM 10. *Let $m \in \mathbb{N}$ and $\delta, \lambda, \mu \in \mathbb{C}$ $(\delta \neq 0)$ and let roots s_1 and s_2 in (5.4) be such that $s_1 \neq s_2 \neq -k$ for any $k \in \mathbb{N}_0$. Then*

$$G_{m;\,\delta,\mu,\lambda}(x) = \frac{(-1)^m}{\delta} \sum_{k=0}^{m-1} \frac{1}{(s_1+k)(s_2+k)(m-k-1)!} \frac{(-x)^k}{k!}$$

$$+ \frac{(-1)^m}{\delta(s_1-s_2)} \left[\frac{\Gamma(s_1)}{\Gamma(s_1+m)} x^{-s_1} - \frac{\Gamma(s_2)}{\Gamma(s_2+m)} x^{-s_2} \right]. \tag{5.6}$$

The particular solution $y(x)$ of equation (1.4) is given by (5.1) and (5.2) provided that the integrals in the right-hand sides of (5.1) and (5.2) are convergent.

THEOREM 11. *Let $m \in \mathbb{N}$ and $\delta, \lambda, \mu \in \mathbb{C}$ $(\delta \neq 0)$ and let roots s_1 and s_2 in (5.4) be such that $s_1 = s_2 \neq -k$ for any $k \in \mathbb{N}_0$. Then*

$$G_{m;\,\delta,\mu,\lambda}(x) = \frac{(-1)^m}{\delta} \sum_{k=0}^{m-1} \frac{1}{(s_1+k)^2(m-k-1)!} \frac{(-x)^k}{k!}$$

$$+ \frac{(-1)^m \Gamma(s_1)}{\delta \Gamma(s_1+m)} x^{-s_1} [\psi(s_1) - \psi(s_1+m) - \log(x)]. \tag{5.7}$$

The particular solution $y(x)$ of equation (1.4) is given by (5.1) and (5.2) provided that the integrals in the right-hand sides of (5.1) and (5.2) are convergent.

Using the relation (3.15) with $\alpha = m \in \mathbb{N}$, from Theorems 3 and 4 we obtain the following assertions.

THEOREM 12. *Let $m \in \mathbb{N}$ and $\delta, \lambda, \mu \in \mathbb{C}$ $(\delta \neq 0)$ and let roots s_1 and s_2 in (5.4) be such that there exists $n_0 \in \mathbb{N}_0$ such that $s_1 \neq s_2$, $s_1 = -n_0$. Then*

$$G_{m;\,\delta,\mu,\lambda}(x) = \frac{1}{\delta} \sum_{k=0, k\neq n_0}^{m-1} \frac{(-1)^m(-x)^k}{k!(k-n_0)(k+s_2)(m-k-1)!} + \frac{(-1)^m \Gamma(s_2)}{\delta \Gamma(s_2+m)} \frac{x^{-s_2}}{(s_2+n_0)}$$

$$+ \frac{(-1)^{n_0+m} x^{n_0}}{\delta(n_0+s_2) n_0! \Gamma(m-n_0)} \left[\gamma + \psi(m-n_0) - \frac{1}{n_0+s_2} + \sum_{j=0}^{n_0-1} \frac{1}{j-n_0} + \log(x) \right]. \tag{5.8}$$

The particular solution $y(x)$ of equation (1.4) is given by (5.1) and (5.2) provided that the integrals in the right-hand sides of (5.1) and (5.2) are convergent.

Theorem 13. *Let $m \in \mathbb{N}$, $\delta, \lambda, \mu \in \mathbb{C}$ $(\delta \neq 0)$ and let roots s_1, s_2 in (5.4) be such that there exists $n_0 \in \mathbb{N}_0$ such that $s_1 \neq s_2$, $s_2 = -n_0$. Then*

$$G_{m;\,\delta,\mu,\lambda}(x) = \frac{1}{\delta} \sum_{k=0, k\neq n_0}^{m-1} \frac{(-1)^m(-x)^k}{k!(k-n_0)(k+s_1)(m-k-1)!} + \frac{(-1)^m \Gamma(s_1)}{\delta \Gamma(s_1+m)} \frac{x^{-s_1}}{(s_1+n_0)}$$

$$+ \frac{(-1)^{n_0+m} x^{n_0}}{\delta(n_0+s_1) n_0! \Gamma(m-n_0)} \left[\gamma + \psi(m-n_0) - \frac{1}{n_0+s_1} + \sum_{j=0}^{n_0-1} \frac{1}{j-n_0} + \log(x) \right]. \tag{5.9}$$

The particular solution $y(x)$ of equation (1.4) is given by (5.1) and (5.2) provided that the integrals in the right-hand sides of (5.1) and (5.2) are convergent.

The next assertion follows from Theorem 5, if we take into account the relations (3.20) and (3.21) with $\alpha = m \in \mathbb{N}$.

Theorem 14. *Let $m \in \mathbb{N}$, $\delta, \lambda, \mu \in \mathbb{C}$ $(\delta \neq 0)$ and let roots s_1 and s_2 in (5.4) be such that there exists $n_0 \in \mathbb{N}_0$ such that $s_1 = s_2 = -n_0$. Then*

$$G_{m;\,\delta,\mu,\lambda}(x) = \frac{(-1)^m}{\delta} \sum_{k=0, k\neq n_0}^{m-1} \frac{(-x)^k}{k!(n_0-k)^2 \Gamma(m-k)}$$

$$+ \frac{(-1)^{n_0+m}}{2\delta n_0! \Gamma(m-n_0)}$$

$$\times \left(\left[\gamma + \psi(m-n_0) + \sum_{j=0}^{n_0-1} \frac{1}{j-n_0} \right]^2 + \psi'(1) - \psi'(m-n_0) + \sum_{j=0}^{n_0-1} \frac{1}{(j-n_0)^2} \right) x^{n_0}$$

$$+ \frac{(-1)^{n_0+m}}{\delta n_0! \Gamma(m-n_0)} \left[\gamma + \psi(m-n_0) + \sum_{j=0}^{n_0-1} \frac{1}{j-n_0} \right] x^{n_0} \log(x)$$

$$+ \frac{(-1)^{n_0+m}}{2\delta n_0! \Gamma(m-n_0)} x^{n_0} \log^2(x). \tag{5.10}$$

The particular solution $y(x)$ of equation (1.4) is given by (5.1) and (5.2) provided that the integrals in the right-hand sides of (5.1) and (5.2) are convergent.

Next, consider special cases of equation (1.4) with $\delta = 0$ and $\lambda = 0$:

$$\mu x^{m+1} y^{(m+1)}(x) + \lambda x^m y^{(m)}(x) = f(x) \quad (x > 0; \quad m \in \mathbb{N}) \tag{5.11}$$

and

$$\delta x^{m+2} y^{(m+2)}(x) + \mu x^{m+1} y^{(m+1)}(x) = f(x) \quad (x > 0; \quad m \in \mathbb{N}). \tag{5.12}$$

Solutions (4.5), (4.6) and (4.14), (4.15) take the respective forms

$$y(x) = \int_x^\infty G_{m;\,\mu,\lambda}\left(\frac{x}{t}\right) f(t) \frac{dt}{t}, \tag{5.13}$$

$$y(x) = \int_1^\infty G_{m;\,\mu,\lambda}\left(\frac{1}{t}\right) f(xt) \frac{dt}{t}, \tag{5.14}$$

and

$$y(x) = \int_x^\infty G_{m;\,\delta,\mu}\left(\frac{x}{t}\right) f(t) \frac{dt}{t}, \tag{5.15}$$

$$y(x) = \int_1^\infty G_{m;\,\delta,\mu}\left(\frac{1}{t}\right) f(xt) \frac{dt}{t}. \tag{5.16}$$

Using (5.5) and (1.8), from Theorems 6 and 7 we deduce the corresponding results for equation (5.11).

THEOREM 15. *Let $m \in \mathbb{N}$ and $\mu, \lambda \in \mathbb{C}$ $(\mu \neq 0, \lambda \neq 0)$ be such that $\lambda \neq \mu(m-n)$ for any $n \in \mathbb{N}_0$. Then*

$$G_{m;\,\mu,\lambda}(x)$$

$$= \frac{(-1)^{m+1}\Gamma(\frac{\lambda}{\mu} - m)}{\mu\Gamma(\frac{\lambda}{\mu})} x^{m-\frac{\lambda}{\mu}} + (-1)^m \sum_{k=0}^{m-1} \frac{(-x)^k}{k!(m-k-1)!(\mu k - \mu m + \lambda)}. \tag{5.17}$$

The particular solution $y(x)$ of equation (5.11) is given by (5.13) and (5.14) provided that the integrals in the right-hand sides of (5.13) and (5.14) are convergent.

THEOREM 16. *Let $m \in \mathbb{N}$ and $\mu, \lambda \in \mathbb{C}$ $(\mu \neq 0, \lambda \neq 0)$ be such that there exists $n_0 \in \mathbb{N}_0$ such that $\lambda = \mu(m - n_0)$. Then*

$$G_{m;\,\mu,\lambda}(x) = \frac{(-1)^{m+1}}{\mu} \sum_{k=0, k\neq n_0}^{m-1} \frac{(-x)^k}{k!(n_0-k)(m-k-1)!}$$

$$+\frac{(-1)^{n_0+m}x^{n_0}}{n_0!\mu\Gamma(m-n_0)}\left[\gamma+\psi(m-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}+\log(x)\right]. \tag{5.18}$$

The particular solution $y(x)$ of equation (5.11) is given by (5.13) and (5.14) provided that the integrals in the right-hand sides of (5.13) and (5.14) are convergent.

Similarly, by (5.5) and (1.8), Theorems 8 and 9 yield the corresponding results for equation (5.12).

THEOREM 17. *Let $m \in \mathbb{N}$ and $\lambda, \mu \in \mathbb{C}$ $(\lambda \neq 0, \mu \neq 0)$ be such that $\mu \neq \delta(m+1-n)$ for any $n \in \mathbb{N}_0$. Then*

$$G_{m;\,\delta,\mu}(x)$$

$$= (-1)^m\frac{\Gamma(-1-m+\frac{\mu}{\delta})}{\delta\Gamma(\frac{\mu}{\delta})}x^{m-\frac{\mu}{\delta}+1}-(-1)^m\sum_{k=0}^{m}\frac{(-x)^k}{k!(m-k)!(\delta k-\delta m-\delta+\mu)}. \tag{5.19}$$

The particular solution $y(x)$ of equation (5.12) is given by (5.15) and (5.16) provided that the integrals in the right-hand sides of (5.15) and (5.16) are convergent.

THEOREM 18. *Let $m \in \mathbb{N}$ $\lambda, \mu \in \mathbb{C}$ $(\lambda \neq 0, \mu \neq 0)$ be such that there exists $n_0 \in \mathbb{N}_0$ such that $\mu = \delta(m+1-n_0)$. Then*

$$G_{m;\,\delta,\mu}(x)=\frac{(-1)^m}{\delta}\sum_{k=0,k\neq n_0}^{m}\frac{(-x)^k}{k!(n_0-k)(m-k)!}$$

$$+\frac{(-1)^{n_0+m}x^{n_0}}{n_0!\delta\Gamma(m+1-n_0)}\left[\gamma+\psi(m+1-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}+\log(x)\right]. \tag{5.20}$$

The particular solution $y(x)$ of equation (5.12) is given by (5.15) and (5.16) provided that the integrals in the right-hand sides of (5.15) and (5.16) are convergent.

Finally, we note that the results of Theorems 10 and 11 stay true for limiting case of equation (1.4) with $\alpha = 0$:

$$\delta x^2 y''(x) + \mu x y'(x) + \lambda y(x) = f(x) \quad (x > 0). \tag{5.21}$$

In this case relations (5.1)–(5.4) take the forms

$$y(x) = \int_x^\infty G_{\delta,\mu,\lambda}\left(\frac{x}{t}\right) f(t)\frac{dt}{t}, \tag{5.22}$$

$$y(x) = \int_1^\infty G_{\delta,\mu,\lambda}\left(\frac{1}{t}\right) f(xt)\frac{dt}{t}, \tag{5.23}$$

$$G_{\delta,\mu,\lambda}(x) = \frac{1}{2\pi i\delta}\int_{c-i\infty}^{c+i\infty} \frac{1}{(s-s_1)(s-s_2)} x^{-s} ds \qquad (c = \Re(s)),$$

$$s_{1,2} = \frac{-\delta+\mu\pm\sqrt{D}}{2\delta}, \quad D = (\delta-\mu)^2 - 4\delta\lambda. \tag{5.24}$$

Then Theorems 10 and 11 yield the following results.

THEOREM 19. *Let $\delta, \lambda, \mu \in \mathbb{C}$ $(\delta \neq 0)$ and let roots s_1 and s_2 in (5.24) be such that $s_1 \neq s_2$. Then*

$$G_{\delta,\mu,\lambda}(x) = \frac{1}{\delta(s_1-s_2)}\left[x^{-s_1} - x^{-s_2}\right]. \tag{5.25}$$

The particular solution $y(x)$ of equation (5.21) is given by (5.22) and (5.23) provided that the integrals in the right-hand sides of (5.22) and (5.23) are convergent.

THEOREM 20. *Let $\delta, \lambda, \mu \in \mathbb{C}$ $(\delta \neq 0)$and let roots s_1 and s_2 in (5.24) coincide: $s_1 = s_2$. Then*

$$G_{\delta,\mu,\lambda}(x) = -\frac{1}{\delta}x^{-s_1}\log(x). \tag{5.26}$$

The particular solution $y(x)$ of equation (5.21) is given by (5.22) and (5.23) provided that the integrals in the right-hand sides of (5.22) and (5.23) are convergent.

REMARK 3. The results of Theorems 15 and 17 stay true in the case $m = 0$ of equations (5.11) and (5.12):

$$\mu x y'(x) + \lambda y(x) = f(x), \quad \delta x^2 y''(x) + \mu x y'(x) = f(x) \quad (x > 0),$$

respectively.

6. Examples

EXAMPLE 1. Consider equation (1.1) with $\alpha = \frac{1}{2}$ and $\delta \neq 0$:

$$\delta x^{\frac{5}{2}}\left(D_-^{\frac{5}{2}}y\right)(x) + \mu x^{\frac{3}{2}}\left(D_-^{\frac{3}{2}}y\right)(x) + \lambda x^{\frac{1}{2}}\left(D_-^{\frac{1}{2}}y\right)(x) = f(x) \quad (x > 0). \tag{6.1}$$

Here the roots s_1 and s_2 in (2.8) take the form

$$s_{1,2} = \frac{-2\delta - \mu \pm \sqrt{D}}{2\delta}, \quad D = (2\delta + \mu)^2 - \delta(3\delta + 2\mu + 4\lambda).$$

If $s_1 \neq s_2 \neq -k$ for any $k \in \mathbb{N}_0$, then by Theorem 1

$$G_{\frac{1}{2};\,\delta,\mu,\lambda}(x) = \frac{1}{\delta}\,{}_2\Psi_3\left[\begin{matrix}(s_1,1), & (s_2,1) & \\ (\frac{1}{2},-1), & (s_1+1,1), & (s_2+1,1)\end{matrix}\Big| -x\right]$$

$$+\frac{1}{\sqrt{D}}\left[\frac{\Gamma(s_1)}{\Gamma(s_1+\frac{1}{2})}x^{-s_1} - \frac{\Gamma(s_2)}{\Gamma(s_2+\frac{1}{2})}x^{-s_2}\right]$$

$$= \frac{1}{\delta\sqrt{\pi}s_1 s_2}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}, s_1, s_2\\ s_1+1, s_2+1\end{matrix}\Big| x\right] + \frac{1}{\sqrt{D}}\left[\frac{\Gamma(s_1)x^{-s_1}}{\Gamma(s_1+\frac{1}{2})} - \frac{\Gamma(s_2)x^{-s_2}}{\Gamma(s_2+\frac{1}{2})}\right]. \tag{6.2}$$

The particular solution of (6.1) is given by

$$y(x) = \int_x^\infty G_{\frac{1}{2};\,\delta,\mu,\lambda}\left(\frac{x}{t}\right) f(t)\frac{dt}{t} = \int_1^\infty G_{\frac{1}{2};\,\delta,\mu,\lambda}\left(\frac{1}{t}\right) f(xt)\frac{dt}{t}. \tag{6.3}$$

Example 2. Consider equation (4.1) with $\alpha = \frac{1}{2}$:

$$\mu x^{\frac{3}{2}}\left(D_-^{\frac{3}{2}}y\right)(x) + \lambda x^{\frac{1}{2}}\left(D_-^{\frac{1}{2}}y\right)(x) = f(x) \quad (x > 0). \tag{6.4}$$

By Theorem 6 and 7 if $\lambda \neq \mu(n - \frac{1}{2})$ for any $n \in \mathbb{N}_0$, then

$$G_{\frac{1}{2};\mu,\lambda}(x) = \frac{\Gamma(-\frac{1}{2}-\frac{\lambda}{\mu})}{\mu\Gamma(-\frac{\lambda}{\mu})}x^{\frac{\lambda}{\mu}+\frac{1}{2}} - {}_1\Psi_2\left[\begin{matrix}(-\frac{1}{2}\mu-\lambda,\mu)\\ (\frac{1}{2},-1),(1-\frac{1}{2}\mu-\lambda,\mu)\end{matrix}\Big| -x\right], \tag{6.5}$$

while if there exists $n_0 \in \mathbb{N}_0$ such that $\lambda = \mu(n_0 - \frac{1}{2})$, then

$$G_{\frac{1}{2};\mu,\lambda}(x) = \frac{1}{\mu}\sum_{k=0}^{n_0-1}\frac{(-1)^k x^k}{k!(n_0-k)\Gamma(\frac{1}{2}-k)} + \frac{(-1)^{n_0}x^{n_0+1}}{\mu}$$

$$\times\,{}_2\Psi_3\left[\begin{matrix}(1,1), & (1,1) & \\ (n_0+2,1), & (2,1), & (\frac{1}{2}-n_0-1,-1)\end{matrix}\Big| -x\right]$$

$$-\frac{(-1)^{n_0}x^{n_0}}{n_0!\mu\Gamma(\frac{1}{2}-n_0)}\left[\gamma+\psi(\frac{1}{2}-n_0)+\sum_{j=0}^{n_0-1}\frac{1}{j-n_0}+\log(x)\right]. \tag{6.6}$$

The particular solution of (6.2) is given by

$$y(x) = \int_x^\infty G_{\frac{1}{2};\mu,\lambda}\left(\frac{x}{t}\right) f(t)\frac{dt}{t} = \int_1^\infty G_{\frac{1}{2};\mu,\lambda}\left(\frac{1}{t}\right) f(xt)\frac{dt}{t}. \tag{6.7}$$

In particular, the equation

$$\frac{2}{3}x^{\frac{3}{2}}\left(D_-^{\frac{3}{2}}y\right)(x) + x^{\frac{1}{2}}\left(D_-^{\frac{1}{2}}y\right)(x) = f(x) \quad (x > 0) \tag{6.8}$$

has the particular solution (6.7) with $\mu = \frac{2}{3}$, $\lambda = 1$ and

$$G_{\frac{1}{2};\frac{2}{3},1}(x) = \frac{3}{2}x^3\,{}_2\Psi_3\left[\begin{matrix}(1,1), \quad (1,1)\\ (4,1), (2,1), (-\frac{5}{2}, -1)\end{matrix}\Big| -x\right]$$

$$-\frac{3x^2}{4\Gamma(-\frac{3}{2})}\left[\gamma + \psi\left(-\frac{3}{2}\right) - \frac{3}{2} + \log(x)\right] + \frac{3}{4\Gamma(\frac{1}{2})} - \frac{3x}{2\Gamma(-\frac{1}{2})}. \tag{6.9}$$

EXAMPLE 3. Consider equation (4.10) with $\alpha = \frac{1}{2}$:

$$\delta x^{\frac{5}{2}}\left(D_-^{\frac{5}{2}}y\right)(x) + \mu x^{\frac{3}{2}}\left(D_-^{\frac{3}{2}}y\right)(x) = f(x) \quad (x > 0). \tag{6.10}$$

By Theorem 8 and 9, if $\mu \neq \delta(n - 1 - \alpha)$ for any $n \in \mathbb{N}_0$,

$$G_{\frac{1}{2};\delta,\mu}(x) = \frac{\Gamma(-\frac{3}{2} - \frac{\mu}{\delta})}{\delta\Gamma(-\frac{\mu}{\delta})}x^{\frac{3}{2}+\frac{\mu}{\delta}} - {}_1\Psi_2\left[\begin{matrix}(-\frac{3}{2}\delta - \mu, \delta)\\ (\frac{3}{2}, -1), (-\frac{3}{2}\delta - \mu + 1, \delta)\end{matrix}\Big| -x\right], \tag{6.11}$$

while if there exists n_0 such that $\mu = \delta(n_0 - 1 - \alpha)$, then

$$G_{\frac{1}{2};\delta,\mu}(x) = \frac{1}{\delta}\sum_{k=0}^{n_0-1}\frac{(-1)^k x^k}{k!(n_0 - k)\Gamma(\frac{3}{2} - k)}$$

$$+\frac{(-1)^{n_0}x^{n_0+1}}{\delta}\,{}_2\Psi_3\left[\begin{matrix}(1,1), \quad (1,1)\\ (n_0 + 2, 1), \ (2,1), \ (\frac{1}{2} - n_0, -1)\end{matrix}\Big| -x\right]$$

$$-\frac{(-1)^{n_0}x^{n_0}}{n_0!\delta\Gamma(\frac{3}{2} - n_0)}\left[\gamma + \psi(\frac{3}{2} - n_0) + \sum_{j=0}^{n_0-1}\frac{1}{j - n_0} + \log(x)\right]. \tag{6.12}$$

The particular solution of (6.10) is given by

$$y(x) = \int_x^\infty G_{\frac{1}{2};\delta,\mu}\left(\frac{x}{t}\right) f(t)\frac{dt}{t} = \int_1^\infty G_{\frac{1}{2};\delta,\mu}\left(\frac{1}{t}\right) f(xt)\frac{dt}{t}. \tag{6.13}$$

In particular, the equation

$$2x^{\frac{5}{2}}\left(D_{-}^{\frac{5}{2}}y\right)(x)+x^{\frac{3}{2}}\left(D_{-}^{\frac{3}{2}}y\right)(x)=f(x)\quad (x>0) \tag{6.14}$$

has the solution (6.13) with $\delta=2$ and $\mu=1$, where

$$G_{\frac{1}{2};2,1}(x)=\frac{1}{2}\left(\frac{1}{2\Gamma(\frac{3}{2})}-\frac{x}{\Gamma(\frac{1}{2})}+x^{3}\,{}_{2}\Psi_{3}\left[\begin{matrix}(1,1),(1,1)\\(2,1),(4,1),(-\frac{3}{2},-1)\end{matrix}\Big|\,-x\right]\right)$$
$$-\frac{x^{2}}{4\Gamma(-\frac{1}{2})}\left(\gamma+\psi(-\frac{1}{2})-\frac{3}{2}+\log(x)\right). \tag{6.15}$$

Example 4. Consider equation (5.11) with $m=2$:

$$\mu x^{3}y'''(x)+\lambda x^{2}y''(x)=f(x)\quad (x>0). \tag{6.16}$$

By Theorems 15 and 16,

$$G_{2;\,\mu,\lambda}(x)=-\frac{\Gamma(-2+\frac{\lambda}{\mu})}{\mu\Gamma(\frac{\lambda}{\mu})}x^{-\frac{\lambda}{\mu}+2}+\frac{1}{\lambda-2\mu}-\frac{x}{\lambda-\mu}, \tag{6.17}$$

if $\lambda\neq\mu(2-n)$ for any $n\in\mathbb{N}_0$, while

$$G_{2;\,\mu,\lambda}(x)=\frac{(-1)^{n_0}x^{n_0}}{n_0!\mu\Gamma(2-n_0)}[\psi(2-n_0)+\log(x)], \tag{6.18}$$

if there exists $n_0\in\mathbb{N}_0$ such that $\lambda=\mu(2-n_0)$. The particular solution $y(x)$ of equation (6.16) is given by

$$y(x)=\int_x^{\infty}G_{2;\,\mu,\lambda}\left(\frac{x}{t}\right)f(t)\frac{dt}{t}=\int_1^{\infty}G_{2;\,\mu,\lambda}\left(\frac{1}{t}\right)f(xt)\frac{dt}{t}. \tag{6.19}$$

In particular, the equation

$$\frac{1}{2}x^{3}y'''(x)+\frac{1}{3}x^{2}y''(x)=f(x)\quad (x>0) \tag{6.20}$$

has the solution

$$y(x)=\int_x^{\infty}G_{2;\,\frac{1}{2},\frac{1}{3}}\left(\frac{x}{t}\right)f(t)\frac{dt}{t}=\int_1^{\infty}G_{2;\,\frac{1}{2},\frac{1}{3}}\left(\frac{1}{t}\right)f(xt)\frac{dt}{t}, \tag{6.21}$$

where

$$G_{2;\,\frac{1}{2},\frac{1}{3}}(x)=-\frac{9}{2}x^{\frac{4}{3}}+6x-\frac{3}{2}. \tag{6.22}$$

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